TRINITY COLLEGE

IN NOMINE DOMINI

**Semester One Examination, 2022**

**Question/Answer booklet**

# MATHEMATICS METHODS UNIT 3

## Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

WA student number: In figures

In words

Your name

**Time allowed for this section**

Reading time before commencing work: five minutes

Working time: fifty minutes

Number of additional answer booklets used (if applicable):

**Materials required/recommended for this section**

***To be provided by the supervisor***

This Question/Answer booklet

Formula sheet

***To be provided by the candidate***

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

**Important note to candidates**

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 7 | 7 | 50 | 52 | 35 |
| Section Two: Calculator-assumed | 12 | 12 | 100 | 95 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free 35% (52 Marks)
This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

---

This page has been left intentionally blank

**Question 1** **(6 marks)**

(a) Determine $f'(-1)$ when $f(x) = 3(2x+1)^4$. (3 marks)

| Solution |
|---|
| $f'(x) = 3(4)(2)(2x+1)^3$<br>$= 24(2x+1)^3$<br><br>$f'(-1) = 24(-1)^3$<br>$= -24$ |
| **Specific behaviours** |
| ✓ recognises the need to use chain rule<br>✓ obtains correct derivative<br>✓ obtains correct value |

(b) Determine $g(3)$ when $g'(x) = 6e^{2x-4}$ and $g(2) = 5$. (3 marks)

| Solution |
|---|
| $g(3) = g(2) + \int_2^3 g'(x)\,dx$<br>$= 5 + \int_2^3 6e^{2x-4}\,dx$<br>$= 5 + [3e^{2x-4}]_2^3$<br>$= 5 + 3e^2 - 3e^0$<br>$= 2 + 3e^2$ |
| **Specific behaviours** |
| ✓ indicates total change is integral of rate of change<br>✓ obtains correct antiderivative<br>✓ obtains correct value |

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

**Question 2** **(7 marks)**

Let $f(x) = 15 - 4x - 6x^2 - 4x^3 - x^4$.

(a) The curve $y = f(x)$ cuts the horizontal axis at $x = 1$. State, with reasons, whether the function is increasing, decreasing or neither at this point. (2 marks)

| Solution |
|---|
| $f'(x) = -4 - 12x - 12x^2 - 4x^3, \qquad f(1) = -4 - 12 - 12 - 4 = -32$<br><br>Since the gradient at this point is negative, then the function is decreasing. |
| **Specific behaviours** |
| ✓ indicates that $f'(1) < 0$<br>✓ uses sign of derivative to deduce function is decreasing |

(b) Determine $f''(0)$ and use this value to describe the concavity of the curve $y = f(x)$ where it crosses the vertical axis. (2 marks)

| Solution |
|---|
| $f''(x) = -12 - 24x - 12x^2, \qquad f(0) = -12$<br><br>The curve is concave down at this point. |
| **Specific behaviours** |
| ✓ correctly evaluates $f''(0)$<br>✓ states concavity |

(c) Does the curve $y = f(x)$ have any points of inflection? If it does, determine the coordinates of their location. If not, justify your answer. (3 marks)

<table>
<tr><th colspan="2">Solution</th></tr>
<tr><td colspan="2">No, the curve does not have any points of inflection.<br><br>$f''(x) = -12(x^2 + 2x + 1) = -12(x + 1)^2, \qquad f''(x) = 0 \Rightarrow x = -1$<br><br>Possible point of inflection at $x = -1$, so test for inflection:</td></tr>
<tr><td>$f''(-1.1) = 12(-1.1 + 1)^2 > 0$<br>$f''(-0.9) = 12(-0.9 + 1)^2 > 0$<br>As curve is concave up on either side of $x = -1$, then not a point of inflection.</td><td>$f'''(x) = -24 - 24x$<br>$f'''(-1) = 0$<br>Since both second and third derivatives are zero at $x = -1$ then not a point of inflection.</td></tr>
<tr><th colspan="2">Specific behaviours</th></tr>
<tr><td colspan="2">✓ states no, with reasonable attempt to justify<br>✓ locates point where $f''(x) = 0$<br>✓ checks concavity either side of point, uses third derivative test, or other valid reasoning</td></tr>
</table>

**Question 3** **(8 marks)**

When a seed is randomly selected from a packet and grown, the probability that it yields a white flower is $0.35 = \frac{7}{20}$.

(a) Explain why this context is suitable for modelling with a Bernoulli random variable and state the mean of the Bernoulli distribution. (2 marks)

| Solution |
|---|
| There is a two-outcome situation (seed yields a white flower or it doesn't).<br><br>$\mu = 0.35$ |
| **Specific behaviours** |
| ✓ indicates a two-outcome situation<br>✓ correct mean |

(b) When several Bernoulli trials are repeated, the total number of successes can be modelled with a binomial random variable provided the trials meet two conditions. Briefly describe these conditions. (2 marks)

| Solution |
|---|
| (i) The trials are independent of each other.<br>(ii) The probability of success is the same for each trial. |
| **Specific behaviours** |
| ✓ states trials independent<br>✓ states trails have same probability |

(c) Nine seeds are randomly selected and grown. Write an expression for the probability that seven or eight of these seeds will yield a white flower. Do not evaluate. (2 marks)

| Solution |
|---|
| $p = \binom{9}{7}(0.35)^7(0.65)^2 + \binom{9}{8}(0.35)^8(0.65)$ |
| **Specific behaviours** |
| ✓ one correct term of expression<br>✓ correct expression |

(d) When a gardener wants to be at least $99\%$ certain of obtaining one or more white flowers, the number of seeds $n$ that must be selected and grown will be the solution of the inequality $b^n \leq a$. State, with justification, the value of the constant $a$ and the value of the constant $b$. (2 marks)

| Solution |
|---|
| $P(X \geq 1) = 1 - P(X = 0)$<br>$= 1 - 0.65^n$<br>Hence<br>$1 - 0.65^n \geq 0.99, \quad 0.65^n \leq 0.01 \quad \rightarrow \quad a = 0.01, b = 0.65$ |
| **Specific behaviours** |
| ✓ indicates correct expression for $P(X \geq 1)$<br>✓ correct values for $a$ and $b$ |

**Question 4** **(8 marks)**

The function $f$ is defined for $x > 0$ by $f(x) = \dfrac{e^{3x-2}}{x}$, and $f''(x) = \dfrac{(9x^2 - 6x + 2)e^{3x-2}}{x^3}$.

(a) Determine the coordinates and nature of all stationary points of the graph of $y = f(x)$. Justify your answer. (6 marks)

**Solution**

$$f'(x) = \frac{(3e^{3x-2})(x) - (1)(e^{3x-2})}{x^2}$$

$$f'(x) = 0 \to e^{3x-2}(3x-1) = 0 \to x = {}^1\!/_3$$

$$f''({}^1\!/_3) = \frac{(1-2+2)e^{-1}}{({}^1\!/_3)^3} = \frac{27}{e}$$

$$f''({}^1\!/_3) > 0 \to \text{stationary point is a minimum}$$

$$f({}^1\!/_3) = \frac{e^{-1}}{{}^1\!/_3} = \frac{3}{e}$$

The only stationary point of the graph is a minimum at $({}^1\!/_3, {}^3\!/_e)$.

**Specific behaviours**

✓ attempts to use quotient rule
✓ correctly obtains $f'(x)$
✓ uses $f(x) = 0$ to determine $x$-coordinate of stationary point
✓ identifies sign of second derivative at stationary point
✓ correctly identifies nature of stationary point
✓ correct coordinates of stationary point

(b) Determine the gradient of $f'(x)$ when $f(x) = \frac{e^4}{2}$. (2 marks)

**Solution**

$$f(2) = \frac{e^4}{2}$$

$$f''(2) = \frac{(9(2)^2 - 6(2) + 2)e^{3(2)-2}}{(2)^3}$$

$$f''(2) = \frac{13e^4}{4}$$

**Specific behaviours**

✓ establishes that $x = 2$
✓ substitutes $x = 2$ into $f''(x)$ and calculates $f''(2) = \frac{13e^4}{4}$

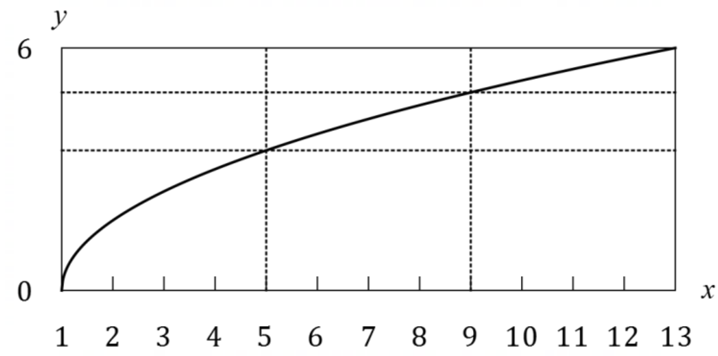
DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

**Question 5** **(8 marks)**

The graph of $y = \sqrt{3x-3}$ between $x = 1$ and $x = 13$ is shown at right.

Approximate values for $\sqrt{12}$ and $\sqrt{24}$ are $3.4$ and $4.9$ respectively.




(a) Use the areas of the rectangles shown to explain why $33.2 < \int_1^{13} \sqrt{3x-3}\,dx < 57.2$. (3 marks)

| Solution |
|---|
| The value of the integral is the area under the curve between $1$ and $13$. The area of the inscribed rectangles is $4(0 + 3.4 + 4.9) = 33.2$, an underestimate. The area of the circumscribed rectangles is $4(3.4 + 4.9 + 6) = 57.2$, an overestimate. Hence the value of the integral must lie between these two. |
| **Specific behaviours** |
| ✓ derives area approximation using inscribed rectangles<br>✓ derives area approximation using circumscribed rectangles<br>✓ explains inequality |

(b) Evaluate $\int_1^{13} \sqrt{3x-3}\,dx$. (3 marks)

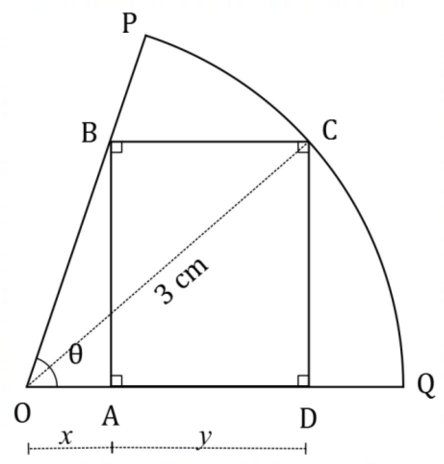
| Solution |
|---|
| $\int_1^{13} (3x-3)^{\frac{1}{2}}\,dx = \left[\frac{2}{9}(3x-3)^{\frac{3}{2}}\right]_1^{13}$<br>$= \frac{2}{9}(36)^{\frac{3}{2}} - \frac{2}{9}(0)^{\frac{3}{2}}$<br>$= \frac{2 \times 6 \times 6 \times 6}{3 \times 3} = 48$ |
| **Specific behaviours** |
| ✓ obtains $(3x-3)^{1.5}$ term in antiderivative<br>✓ obtains correct antiderivative<br>✓ substitutes both bounds and simplifies |

(c) Evaluate $\int_1^{13} \left(3 - \sqrt{3x-3}\right) dx$. (2 marks)

| Solution |
|---|
| $\int_1^{13} \left(3 - \sqrt{3x-3}\right) dx = \int_1^{13} 3\,dx - \int_1^{13} \sqrt{3x-3}\,dx$<br>$= 36 - 48 = -12$ |
| **Specific behaviours** |
| ✓ uses linearity<br>✓ correct value |

**Question 6** **(7 marks)**

The diagram shows the vertices of rectangle $ABCD$ lying on sector $OPQ$ that subtends an angle of $\theta$ in a circle of radius 3 cm, and where $\tan\theta = 3$. Let $OA = x$ cm and $AD = y$ cm.




(a) Show that the perimeter of the rectangle is given by $p = 4x + 6\sqrt{1-x^2}$ cm. (3 marks)

| Solution |
|---|
| $AB = OA\tan\theta = 3x, \qquad CD = 3x$<br><br>$OD^2 + DC^2 = OC^2$<br>$(x+y)^2 + (3x)^2 = 3^2$<br>$(x+y)^2 = 9 - 9x^2$<br>$x + y = 3\sqrt{1-x^2}$<br>$\therefore AD = y = 3\sqrt{1-x^2} - x$<br><br>$p = 2(AB + AD)$<br>$= 2\left(3x + 3\sqrt{1-x^2} - x\right)$<br>$= 4x + 6\sqrt{1-x^2}$ |
| **Specific behaviours** |
| ✓ derives expression for $AB$<br>✓ derives expression for $AD$<br>✓ derives expression for perimeter |

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

(b) Determine the maximum perimeter of rectangle $ABCD$. (4 marks)

| Solution |
|---|
| Derivative of $p$ with respect to $x$: $$\frac{dp}{dx} = 4 + 6\left(\frac{1}{2}\right)(1 - x^2)^{-\frac{1}{2}}(-2x) = 4 - \frac{6x}{\sqrt{1 - x^2}}$$ Derivative will be zero when $\frac{dp}{dx} = 0$. Hence $$6x = 4\sqrt{1 - x^2}$$ $$3x = 2\sqrt{1 - x^2}$$ Squaring both sides: $$9x^2 = 4 - 4x^2$$ $$13x^2 = 4$$ $$x = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \text{ cm}$$ Finally determine perimeter: $$p = 4\left(\frac{2\sqrt{13}}{13}\right) + 6\sqrt{1 - \left(\frac{2\sqrt{13}}{13}\right)^2} = \frac{8\sqrt{13}}{13} + 6\sqrt{\frac{13^2 - 4 \times 13}{13^2}} = \frac{8\sqrt{13}}{13} + 6\sqrt{\frac{9 \times 13}{13^2}} = \frac{8\sqrt{13}}{13} + \frac{18\sqrt{13}}{13} = 2\sqrt{13} \text{ cm}$$ |
| **Specific behaviours** |
| ✓ obtains derivative<br>✓ equates derivative to zero and starts solving equation<br>✓ obtains correct value of $x$<br>✓ obtains correct perimeter |

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

**Question 7** **(8 marks)**

Let $f(x) = e^{-3x}(\cos 3x + \sin 3x)$.

(a) Determine $f'(x)$, simplifying your answer. (3 marks)

| **Solution** |
|---|
| $f'(x) = (-3e^{-3x})(\cos 3x + \sin 3x) + (e^{-3x})(-3\sin 3x + 3\cos 3x)$<br>$= -6e^{-3x}\sin 3x$ |
| **Specific behaviours** |
| ✓ correctly applies product rule<br>✓ correctly differentiates trig terms<br>✓ simplifies to obtain correct derivative |

(b) Use differentiation and your previous answer to show that

$$\int \left(e^{-3x}\sin 3x\right) dx = -\frac{1}{6}e^{-3x}(\cos 3x + \sin 3x) + c,$$

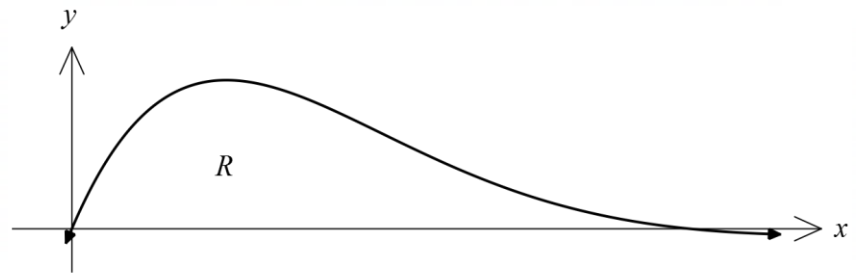
where $c$ is a constant. (2 marks)

| **Solution** |
|---|
| Derivative of LHS (using derivative of integral of a function is original function):<br>$\frac{d}{dx}\left(\int \left(e^{-3x}\sin 3x\right) dx\right) = e^{-3x}\sin 3x$<br><br>Derivative of RHS (using part (a)):<br>$\frac{d}{dx}\left(-\frac{1}{6}e^{-3x}(\cos 3x + \sin 3x) + c\right) = -\frac{1}{6} \times (-6e^{-3x}\sin 3x) = e^{-3x}\sin 3x$<br><br>Hence LHS=RHS. |
| **Specific behaviours** |
| ✓ differentiates LHS<br>✓ differentiates RHS and simplifies to equal derivative of LHS |

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

(c) The graph of $y = e^{-3x}\sin 3x$ is shown below. Determine the area of the region $R$, bounded by the curve and the $x$-axis.

(3 marks)




| Solution |
|---|
| $\sin 3x = 0 \Rightarrow 3x = \pi, x = \frac{\pi}{3}$ <br> $\int_0^{\frac{\pi}{3}} e^{-3x}\sin 3x\,dx = \left[\frac{-e^{-3x}}{6}(\cos 3x + \sin 3x)\right]_0^{\frac{\pi}{3}}$ <br> $= \left(-\frac{e^{-\pi}}{6}(\cos\pi + \sin\pi)\right) - \left(-\frac{e^{0}}{6}(\cos 0 + \sin 0)\right)$ <br> $= \frac{e^{-\pi}}{6} + \frac{1}{6} = \frac{e^{-\pi}+1}{6}$ |
| **Specific behaviours** |
| ✓ forms integral with correct bounds <br> ✓ writes antiderivative and substitutes bounds <br> ✓ simplifies |

**End of questions**

Supplementary page

Question number: __________

Supplementary page

Question number: __________